The resistance to the relative motion of gases

I. P. WILLIAMS and D. K. SARVAJNA

Department of Mathematics, University of Reading

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Abstract. Expressions for the resistance to the motion of one gas through another at any relative drift speed are found, on the basis of a simple kinetic theory model. Approximate expressions giving a good degree of accuracy over all speeds are proposed, and it is suggested that the results will be of use in, for example, the study of gas streaming from stars.

1. Introduction

In many fields of physics and astronomy the problem of the motion of one gas through another is of considerable importance. We mention only the problems associated with ejection of gas from the atmosphere of a star, those involving collision mechanisms in interstellar space and problems associated with cloud formation in the Earth's atmosphere. Rosenberg (1934), in an astronomical context, obtained the resistance of one gas moving through another by assuming that both gases have the same molecular mass and the same kinetic temperature. In this paper, using a somewhat different method of calculation, we generalize the problem to include the case of dissimilar gases with different molecular masses and different kinetic temperatures. No transfer of heat from one gas to the other is considered, however, and we assume that the molecules of both gases can be regarded as maintaining their equilibrium velocity distribution throughout (see § 8). The expressions found reduce to known forms for a number of special cases, which are discussed. Simple approximate forms which give a good degree of accuracy over all relative speeds are also obtained, together with an estimate of the error introduced by using such approximations. Cases when the model would appear to be a good representation of the physical situation are discussed as well.

2. The kinetic model

We assume that the molecules of the two gases can be represented as small elastic spheres. Let gas 1 have n_1 molecules per unit volume, each of mass m_1 and radius a_1, and let gas 2 have n_2 molecules per unit volume, each of mass m_2 and radius a_2. We assume that the random velocities of the molecules of both gases have a Maxwellian distribution about their average drift velocity, this being the equilibrium distribution. Thus the probability of a molecule of gas 1 having a velocity component along some prescribed direction between u_1 and $u_1 + du_1$ relative to the average drift velocity of the molecules of gas 1 is

$$f_1(u_1)\,du_1 = \left(\frac{\beta_1}{\pi}\right)^{1/2} \exp(-\beta_1 u_1{}^2)\,du_1 \tag{1}$$

where $\beta_1 = m_1/2kT_1$, k being Boltzmann's gas constant and T the kinetic temperature of gas 1.

Similarly, the probability of a molecule of gas 2 having a velocity component between u_2 and $u_2 + du_2$ in the same direction, relative to the average drift velocity of the molecules in gas 2, is

$$f_2(u_2)\,du_2 = \left(\frac{\beta_2}{\pi}\right)^{1/2} \exp(-\beta_2 u_2{}^2)\,du_2 \tag{2}$$

where $\beta_2 = m_2/2kT_2$ and T_2 is the kinetic temperature of gas 2.

Let the relative drift velocities of the two gases be $\mathbf{V}$, and let us take the direction of $\mathbf{V}$ as the polar axis in a system of spherical polar coordinates (r, θ, ϕ) with origin at the centre of a

typical molecule of gas 2. In a collision between this molecule and a molecule of gas 1 we take θ to denote the angle between the polar axis and the line of centres of the two molecules at the moment of impact. Further, let us take this line of centres to be the prescribed direction mentioned earlier, so that the molecules have velocity components u_1 and u_2, respectively, along this line.

The reflection of the molecules is assumed to be specular, so that the component of relative velocity along the line of centres is reversed, while no change in the velocity occurs perpendicular to this line. We assume that the effective cross section for collision of the molecules is dependent on their relative velocity $(u_1 + u_2 + V \cos \theta)$. If we take the repulsive force between molecules in collision to be K^1/r^n, where K^1 is a constant and r their separation, the distance of closest approach σ is given by

$$\sigma^2 = A(u_1 + u_2 + V \cos \theta)^{-4/(n-1)} \tag{3}$$

where A is another constant $(= 2K^1/(n-1))$.

For the special case of hard spheres $n \to \infty$ and $\sigma = a_1 + a_2$.

3. Evaluation of the resistance

In a collision between a typical molecule of gas 2 and a typical molecule of gas 1, the transfer of momentum along the line of centres is easily shown to be

$$\frac{2m_1 m_2}{m_1 + m_2}(u_1 + u_2 + V \cos \theta). \tag{4}$$

The total number of such collisions per unit time per unit volume is

$$\sigma^2 n_1 n_2 (u_1 + u_2 + V \cos \theta) f_1(u_1) f_2(u_2) \sin \theta \, d\theta \, d\phi \, du_1 \, du_2. \tag{5}$$

The momentum transferred from one gas to the other per unit time along the line of centres due to such collisions is given, from equations (4) and (5), as

$$dR = \frac{2m_1 n_1 m_2 n_2}{m_1 + m_2} \sigma^2 f_1(u_1) f_2(u_2)(u_1 + u_2 + V \cos \theta) \sin \theta \, d\theta \, d\phi \, du_1 \, du_2 \tag{6}$$

per unit volume of gas 2.

From symmetry considerations the net transfer of momentum will be along the polar axis. Its magnitude will be given by integrating $dR \cos \theta$ over all possible values of the variables θ, ϕ, u_1 and u_2.

Rosenberg (1934) obtained an expression similar to (6), but with both velocity distributions taken relative to the drift velocity of gas 1 and $\sigma = a_1 + a_2$. Since he assumed that $m_1 = m_2$ and $\beta_1 = \beta_2$, a simple transformation of the variables enabled him to evaluate the integral. However, if $m_1 \neq m_2$, the simplicity of this approach fails. In the present calculation the limits of integration have to be chosen with some care, but the integration is straightforward in several cases of interest.

The limits of integration for u_2 can be taken to be $-\infty$ to $+\infty$, and those for ϕ from 0 to 2π. For the other variables, however, not all ranges of values correspond to possible collisions between the molecules. Provided $u_1 > V - u_2$, collisions are possible for all values of the angle θ. But if u_1 satisfies $-V - u_2 < u_1 < V - u_2$, collisions are only possible for $0 < \theta < \alpha$, where

$$\alpha = \cos^{-1}\left(-\frac{u_1 + u_2}{V}\right).$$

If $u_1 < -V - u_2$, no collisions occur. If we insert these limits in the integration the resistance to the relative motion of the gases, being the rate of transfer of momentum, is given by

$$R = \frac{2m_1 n_1 m_2 n_2}{m_1 + m_2} \frac{(\beta_1 \beta_2)^{1/2}}{\pi}(I_1 + I_2) \tag{7}$$

where

$$I_1 = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} du_2 \int_{V-u_2}^{\infty} du_1 \int_0^{\pi} \sigma^2 \sin\theta \, d\theta (u_1+u_2+V\cos\theta)^2$$

$$\times \exp(-\beta_1 u_1{}^2) \exp(-\beta_2 u_2{}^2) \cos\theta$$

and

$$I_2 = \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} du_2 \int_{-V-u_2}^{V-u_2} du_1 \int_0^{\alpha} \sigma^2 \sin\theta \, d\theta (u_1+u_2+V\cos\theta)^2$$

$$\times \exp(-\beta_1 u_1{}^2) \exp(-\beta_2 u_2{}^2) \cos\theta.$$

General evaluation of these integrals with σ substituted from equation (3) is clearly not possible. However, integration is possible for several forms for σ.

4. Exact evaluation of the integrals
4.1. *Case I: hard spheres*
For this case

$$\sigma = \text{const.} = a_1 + a_2.$$

Considering I_1 only and integrating over θ and ϕ, we obtain

$$I_1 = (a_1+a_2)^2 \frac{8\pi V}{3} \int_{-\infty}^{\infty} du_2 \int_{V-u_2}^{\infty} du_1 (u_1+u_2) \exp(-\beta_1 u_1{}^2) \exp(-\beta_2 u_2{}^2).$$

Transforming the variables (u_1, u_2) to (ξ, η) by means of the relations

$$\xi = u_1 + u_2, \qquad \eta = \frac{\beta_2 u_2 - \beta_1 u_1}{\beta_1 + \beta_2}$$

we find that

$$I_1 = \frac{8V(a_1+a_2)^2}{3} \int_{-\infty}^{\infty} \exp\{-(\beta_1+\beta_2)\eta^2\} \, d\eta \int_V^{\infty} \xi \exp(-\beta\xi^2) \, d\xi$$

$$= \tfrac{4}{3}\pi^{3/2} V(a_1+a_2)^2 \frac{\exp(-\beta V^2)}{\beta(\beta_1+\beta_2)^{1/2}}$$

where for convenience we have written

$$\beta = \frac{\beta_1 \beta_2}{\beta_1 + \beta_2}.$$

Similarly for I_2, on integrating over θ and ϕ, we obtain

$$I_2 = 2\pi(a_1+a_2)^2 \int_{-\infty}^{\infty} du_2 \int_{-V-u_2}^{V-u_2} du_1 \left\{ \tfrac{1}{2}(u_1+u_2)^2 + \tfrac{2}{3}(u_1+u_2)V \right.$$

$$\left. + \tfrac{1}{4}V^2 - \frac{(u_1+u_2)^4}{12v^2} \right\} \exp(-\beta_1 u_1{}^2) \exp(-\beta_2 u_2{}^2).$$

Transforming the variables (u_1, u_2) to (ξ, η) by means of the same equations as given above, we obtain

$$I_2 = 2\pi(a_1+a_2)^2 \int_{-\infty}^{\infty} \exp\{-(\beta_1+\beta_2)\eta^2\} \, d\eta \int_{-V}^{V} \left(\frac{\xi^2}{2V} + \frac{2}{3}\xi V + \frac{V^2}{4} - \frac{\xi^4}{12v^2} \right) \exp(-\beta\xi^2) \, d\xi$$

$$= \frac{\pi^{3/2}(a_1+a_2)^2}{(\beta_1+\beta_2)^{1/2}} \left\{ \left(\frac{1}{V\beta} - \frac{10}{3}V \right) \frac{\exp(-\beta V^2)}{4\beta} + \left(4\beta^2 V^2 + 4\beta - \frac{1}{V^2} \right) \frac{\sqrt{\pi}}{8\beta^{3/2}} \text{erf}(\sqrt{\beta}V) \right\}$$

where erf denotes the error function, defined by

$$\text{erf } x = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-t^2) \, dt$$

(see, for example, Magnus and Oberhettinger 1954). Substituting these values for I_1 and I_2 into equation (7), we obtain an expression for the resistance per unit volume on gas 2 due to gas 1, in the form

$$R = \frac{m_1 n_1 m_2 n_2}{m_1 + m_2} (a_1 + a_2)^2 \frac{\sqrt{\pi}}{2\beta} \left\{ \left(\frac{1}{S} + 2S \right) \exp(-S^2) + \left(S^2 + 1 - \frac{1}{4S^2} \right) 2\sqrt{\pi} \text{ erf } S \right\}. \tag{8}$$

For convenience we have written $\sqrt{\beta} V = S$. S is clearly some form of a molecular speed ratio for the gases, as $\sqrt{\beta} = 2/\sqrt{\pi} W$, where W is a mean thermal velocity for the gases, defined by

$$W = 2 \left\{ \frac{2k}{\pi} \left(\frac{T_1}{m_1} + \frac{T_2}{m_2} \right) \right\}^{1/2}.$$

4.2. *Case 2*

Taking $n = 3$ in equation (3), then we have

$$\sigma^2 = A(u_1 + u_2 + V \cos \theta)^{-2}.$$

In this case the integrals to be evaluated become

$$I_1 = A \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} du_2 \int_{V - u_2}^{\infty} du_1 \int_0^{\pi} \sin \theta \cos \theta \exp(-\beta_1 u_1^2) \exp(-\beta_2 u_2^2) \, d\theta$$

which is clearly zero, and

$$I_2 = A \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} du_2 \int_{-V - u_2}^{V - u_2} du_1 \int_0^{\alpha} \sin \theta \cos \theta \exp(-\beta_1 u_1^2) \exp(-\beta_2 u_2^2) \, d\theta.$$

If we integrate this second integral over θ and ϕ and use the same transformation of the variables as before, this integral can be evaluated, to give

$$I_2 = \frac{\pi^{3/2} A}{(\beta_1 + \beta_2)^{1/2}} \left\{ \frac{\exp(-\beta V^2)}{V\beta} - \left(\frac{1}{2\beta V^2} - 1 \right) \left(\frac{\pi}{\beta} \right)^{1/2} \text{erf}(\sqrt{\beta} V) \right\}.$$

Substituting the values of I_1 and I_2 into equation (7) and again writing V in terms of the molecular speed ratio S, we obtain

$$R = \frac{2 m_1 n_1 m_2 n_2 A}{m_1 + m_2} \sqrt{\pi} \left\{ \frac{\exp(-S^2)}{S} + \left(1 - \frac{1}{2S^2} \right) \sqrt{\pi} \text{ erf } S \right\}. \tag{9}$$

4.3. *Case 3*

With $n = 5$, the cross section, from equation (3), is given by

$$\pi \sigma^2 = A \pi (u_1 + u_2 + V \cos \theta)^{-1}.$$

Now, by using transformations that have already been defined, we obtain

$$I_1 = A \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} du_2 \int_{V - u_2}^{\infty} du_1 \int_0^{\pi} \sin \theta V \cos^2 \theta \, d\theta \exp(-\beta_1 u_1^2) \exp(-\beta_2 u_2^2)$$

$$= \frac{2\pi^2 V A}{3(\beta_1 \beta_2)^{1/2}} \left\{ 1 - \text{erf}(\sqrt{\beta} V) \right\}$$

and

$$I_2 = A \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} du_2 \int_{-V-u_2}^{V-u_2} du_1 \int_0^{\alpha} \sin\theta \cos\theta (u_1 + u_2 + V\cos\theta) \exp(-\beta_1 u_1{}^2)$$
$$\times \exp(-\beta_2 u_2{}^2)\, d\theta$$
$$= \frac{2\pi^2 A}{3(\beta_1\beta_2)^{1/2}} V \operatorname{erf}(\sqrt{\beta}V).$$

Substituting these values for I_1 and I_2 into equation (7) gives the resistance in this case as

$$R = \frac{4m_1 n_1 m_2 n_2}{3(m_1 + m_2)} \frac{AS}{\sqrt{\beta}} \tag{10}$$

an expression which is independent of the gas temperature as $S = \sqrt{\beta}V$. The only case that is likely to be of physical interest is the first, namely the hard-sphere case.

5. Approximate expressions

Expressions (8), the one that is physically interesting, is mathematically rather unwieldy and its application in physical problems is inevitably restricted because of this. It is therefore useful to obtain approximate expressions for this, which are more amenable to further mathematical analysis. This can be done by using known expansions for the error function (see Magnus and Oberhettinger 1954), both for small values of S and for large values of S.

Expanding the exponential and the error function in expression (8) for small values of S, we obtain, after simplification,

$$R = 8\frac{m_1 n_1 m_2 n_2}{m_1 + m_2}(a_1 + a_2)^2 \frac{\sqrt{\pi}}{\beta}\left\{\frac{S}{3} + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{S^{2n+1}}{(2n-1)(2n+1)(2n+3)}\right\} \tag{11}$$

while the expansion of the error function for large S gives the resistance in the form

$$R = \frac{m_1 n_1 m_2 n_2}{m_1 + m_2}(a_1 + a_2)^2 \frac{\pi}{\beta}\left\{\left(S^2 + 1 - \frac{1}{4S^2}\right) + \frac{\exp(-S^2)}{2\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(2n)!(n+1)}{(n-1)!2^{n-1}S^{2n+3}}\right\}. \tag{12}$$

Similarly, expression (9) reduces to

$$R = 8\frac{m_1 n_1 m_2 n_2}{m_1 + m_2} A\sqrt{\pi} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{S^{2n+1}}{(2n+1)(2n+3)} \quad \text{(for small } S\text{)} \tag{13}$$

$$R = \frac{2m_1 n_1 m_2 n_2}{m_1 + m_2} A\pi\left[-\frac{1}{2S^2} + 1 + \frac{\exp(-S^2)}{S\sqrt{\pi}}\left\{\frac{1}{S^2} - \sum_{n=1}^{\infty} \frac{(2n)!(n+1)}{2^{n-1}n(n-1)!} \frac{1}{(2S)^{n+1}}\right\}\right] \tag{14}$$

(for large S).

Expression (10) is in a simple form as it stands and requires no further simplification.

It is also possible to evaluate the general integrals given in section 3 for any value of σ in the two limiting cases of large and small translational speed. We shall now discuss this possibility.

5.1. *Small values of the translational velocity V*

In this case the integration over u_1 becomes approximately

$$\int_{-u_2}^{-u_2} du_1$$

which is zero. Hence $I_2 \to 0$. On substituting

$$\pi\sigma^2 = (u_1 + u_2 + V\cos\theta)^{-4/(n-1)} = (u_1 + u_2 + V\cos\theta)^{-\gamma}$$

we find

$$I_1 = \int_0^{2\pi} \frac{A}{\pi} d\phi \int_{-\infty}^{\infty} du_1 \exp(-\beta_2 u_2{}^2) \, du_2 \int_{-u_2}^{\infty} \exp(-\beta_1 u_1{}^2) \, du_1 \int_0^{\pi} (u_1+u_2+V\cos\theta)^{2-\gamma}$$
$$\times \sin\theta\cos\theta \, d\theta.$$

Expanding in powers of $V\cos\theta/(u_1+u_2)$ and ignoring second-order terms, we find, on integrating over ϕ,

$$I_1 = 2A \int_{-\infty}^{\infty} \exp(-\beta_2 u_2{}^2) \, du_2 \int_{-u_2}^{\infty} \exp(-\beta_1 u_1{}^2) \, du_1 \int_0^{\pi} (u_1+u_2)^{2-\gamma} \left(1+\frac{(2-\gamma)V\cos\theta}{u_1+u_2}\right)$$
$$\times \sin\theta\cos\theta \, d\theta$$
$$= \frac{4A(2-\gamma)V}{3} \int_{-\infty}^{\infty} \exp(-\beta_2 u_2{}^2) \, du_2 \int_{-u_2}^{\infty} (u_1+u_2)^{1-\gamma}\exp(-\beta_1 u_1{}^2) \, du_1.$$

Using the same transformation as previously employed, namely $\xi = u_1+u_2$, $\eta = (\beta_2 u_2 - \beta_1 u_1)/(\beta_1+\beta_2)$, we find that

$$I_1 = \frac{4A(2-\gamma)V}{3} \int_{-\infty}^{\infty} \exp\{-(\beta_1+\beta_2)\eta\} \, d\eta \int_0^{\infty} \xi^{1-\gamma}\exp(-\beta\xi^2) \, d\xi$$
$$= \tfrac{2}{3}(2-\gamma)\left(\frac{\pi}{\beta_1+\beta_2}\right)^{1/2} AV\beta^{\frac{1}{2}\gamma-1} \int_0^{\infty} t^{(1-\frac{1}{2}\gamma)-1} e^{-t} \, dt$$
$$= \tfrac{2}{3}(2-\gamma)\left(\frac{\pi}{\beta_1+\beta_2}\right)^{1/2} AV\beta^{\frac{1}{2}\gamma-1}\Gamma(1-\tfrac{1}{2}\gamma) \tag{15}$$

where $\Gamma(a)$ is the gamma function.

The resistance in this case is given by

$$R = \tfrac{4}{3}(2-\gamma)\frac{m_1 n_1 m_2 n_2}{m_1+m_2}\left(\frac{\beta}{\pi}\right)^{1/2} AV\beta^{\frac{1}{2}\gamma-1}\Gamma(1-\tfrac{1}{2}\gamma) \tag{16}$$

on substitution into equation (7).

We note that, when $\gamma = 0$ (hard-sphere case with $A = \pi(a_1+a_2)^2$), equation (16) gives the resistance as

$$\frac{8}{3}\frac{m_1 n_1 m_2 n_2}{m_1+m_2}\left(\frac{\pi}{\beta}\right)^{1/2}(a_1+a_2)^2 V$$

in agreement with equation (11).

Further, with $\gamma = 1$ corresponding to $n = 5$, we find that the resistance is

$$R = \frac{4}{3}\frac{m_1 n_1 m_2 n_2}{m_1+m_2} AV$$

in agreement with equation (10). Because of a singularity at $\gamma = 2$ no expression can be found for the case $n = 3$.

Expression (16) is therefore consistent with the other expression found. A case that is likely to be of interest is that when $n = 13$ (most likely physical value). For this value $\gamma = \tfrac{1}{3}$, and the resistance for small drift velocities is given by

$$R = \frac{20}{9}\frac{m_1 n_1 m_2 n_2}{m_1+m_2}\frac{AV}{\sqrt{\pi}}\beta^{1/3}\Gamma(\tfrac{5}{6}). \tag{17}$$

5.2. *Large values of the translation speed V*

For this case $I_1 \to 0$, and after expanding in terms of $(u_1+u_2)/V\cos\theta$ and ignoring

second-order terms, we find that

$$I_2 = \frac{AV^{2-\gamma}}{\pi} \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} du_2 \int_{-V}^{V} du_1 \int_0^{\pi/2} \exp(-\beta_1 u_1{}^2)$$

$$\times \exp(-\beta_2 u_2{}^2) \left\{ 1 + (2-\gamma) \frac{u_1+u_2}{V \cos \theta} \right\} \sin \theta \cos^{3-\gamma}\theta \, d\theta$$

$$= \frac{2\pi A V^{2-\gamma}}{(4-\gamma)(\beta_1\beta_2)^{1/2}} \qquad (\text{as } \mathrm{erf}(\sqrt{\beta_1}V) \to 1).$$

The resistance is now given by

$$\frac{4n_1 m_1 n_2 m_2}{(4-\gamma)(m_1+m_2)} AV^{2-\gamma}. \tag{18}$$

With $\gamma = 0$, this becomes

$$\frac{\pi(a_1+a_2)^2 m_1 n_1 m_2 n_2}{m_1+m_2} V^2$$

in agreement with equation (12). With $\gamma = 1$, equation (18) becomes

$$R = \frac{4}{3}\frac{\pi m_1 n_1 m_2 n_2}{m_1+m_2}(a_1+a_2)^2 V$$

in agreement with equation (10). $\gamma = 2$ again has a singularity in the integration (integrand becoming independent of V), and so no comparison can be made.

For the physically interesting case when $n = 13$, for high drift velocity, we have

$$R = \frac{12}{11}\frac{m_1 n_1 m_2 n_2}{m_1+m_2} AV^{5/3}. \tag{19}$$

6. Reduction to known forms

In expressions (8) to (14) we have found expressions for the resistance to the relative motion of gases, together with approximations valid for small and high drift velocity. It is interesting to note that some of these reduce under special conditions to expressions that are already known.

If the two gases are identical, then $a_1 = a_2 = a$, $m_1 = m_2 = m$ and $T_1 = T_2 = T$. Hence $\beta_1 = \beta_2 = 2\beta = m/2kT$, and equation (8) reduces to

$$R = 4n_1 n_2 a^2 \sqrt{\pi kT}\left[\left\{\frac{1}{2}\left(\frac{m}{kT}\right)^{1/2}V + \frac{2}{V}\left(\frac{kT}{m}\right)^{1/2}\right\}\exp\left(-\frac{mV^2}{4kT}\right)\right.$$

$$\left. + \left(\frac{mV^2}{4kT}+1-\frac{kT}{mV^2}\right)2\sqrt{\pi}\int_0^{\sqrt{(\pi/2kT)}V}\exp(-x^2)\,dx\right].$$

This is the expression found by Rosenberg (1934) for the resistance between two identical gases when their relative drift velocity is V. Expression (8) thus contains the results of Rosenberg as a special case.

If one of the gases possesses very large molecules, gas 2 say, then $a_2 = a \gg a_1$, $m_2 = m \gg m_1$. In consequence, $\beta_2 \gg \beta_1 = \beta$ and equation (8) gives the resistance on one molecule of gas 2 as

$$R = \frac{\pi^{3/2}mna^2W^2}{4}\left\{\left(S+\frac{1}{2S^2}\right)\exp(-S^2)+\sqrt{\pi}\left(S^2+1-\frac{1}{4S^2}\right)\mathrm{erf}\,S\right\}.$$

This is identical with the expression found by Baines *et al.* (1965) for the resistance to the motion of a sphere moving through a gas.

McCrea (1935) has given expressions for the resistance to the motion of gases in the limiting cases of low and high translation speeds, by assuming that the molecules behave like hard spheres. For low speed he gives

$$R = \tfrac{8}{3} n_1 n_2 (a_1 + a_2)^2 \left(\frac{m_1 m_2}{m_1 + m_2} 2\pi kT \right)^{1/2} V.$$

This is equivalent to taking the first term in the expansion (11) with $T_1 = T_2 = T$ on substituting for S and β.

For high speeds McCrea's formula is

$$R = \pi n_1 n_2 (a_1 + a_2)^2 \frac{m_1 m_2}{m_1 + m_2} V^2$$

which is the first term of (12) with $T_1 = T_2 = T$ and appropriate substitution for S and β carried out. Hence the known expressions for the resistance all appear as special cases of the expressions found above.

7. Error involved in approximating for the hard-sphere case

We have already stated that expressions (8) and (9) are mathematically rather unwieldy and have suggested approximate expressions that are of more practical use. It is of considerable interest to know what error is introduced by using such approximation. As the hard-sphere case is more likely to be of interest in practice, we shall now consider the error introduced when we approximate to the exact expressions by taking the first few terms of the series in (11) and (12). The procedure can easily be carried out for the other case using series (13) and (14) if required. The convergence is in fact quite rapid in both cases. The question has been considered by Baines et al. (1965), where the same function occurs and we shall merely summarize their conclusions here.

Taking only the first term of the expansion in (11), we find that the error is less than 5% for $S < 0.5$, but increases up to 30% as S increases to 1·4. The first two terms of (11), on the other hand, give an error of less than 5% for all $S \leqslant 1.4$. Again, if we take the first term only of the expansion (12) the error is less than 5% while $S > 5$, but rises up to 50% as S decreases to 1. Taking two terms of (12) gives an error of less than 5% for all $S > 1.4$.

This suggests that a practical expression for the resistance in the hard-sphere case would be

$$R = \frac{m_1 n_1 m_2 n_2}{m_1 + m_2} \sigma^2 \beta^{-1} \begin{cases} \tfrac{8}{3}\sqrt{\pi}(S + \tfrac{1}{6}S^3) & (S \leqslant 1.4) \\ \\ \pi(1 + S^2) & (S > 1.4). \end{cases} \qquad (20)$$

In the same way practical expressions for the case $n = 3$ can be found if the need for them arises. The expressions found for the physically interesting case $n = 13$ are already approximations and no simpler practical expressions can be given.

8. Application

The kinetic model we have used is valid only if the equilibrium distribution of velocities remains essentially undisturbed. The restriction also applies to the work of Rosenberg. In general, this is likely to remain true only if the penetration remains less than about one mean free path. There are, however, situations, which we now discuss, in which the equilibrium distribution condition is likely to be satisfied for a reasonable degree for much longer.

If one of the gases is very rarefied, in comparison with the other, such as might occur in the flow of particles through interstellar gas following a nova or supernova explosion, the interaction of molecules of the rarefied gas is negligible and both gases may be assumed to retain their equilibrium velocity distribution. A slightly different case is when a stream of one gas passes through a second gas, the dimensions of the cross section of the stream being small in comparison with the mean free path of the second gas. This situation arises in the study of comet tails in the solar wind and other phenomena involving ejection from the Sun.

Finally, the assumption is valid when the masses and temperatures of the two gases are very close, as in Rosenberg's work, and this corresponds to the situation when two interstellar clouds collide.

9. Conclusion

The expressions (8) and (9) for the resistance to the relative motion of dissimilar gases will be of most interest in situations where the known limiting forms discussed in § 6 for small and large relative drift velocities are too inaccurate, that is mainly at velocities comparable with the value of $\beta^{-1/2}$. Concerning the value of β, we have that for gases with comparable molecular masses at widely different temperatures β will approximate to $m/2kT$ evaluated for the high-temperature gas, while for gases at similar temperatures but with different molecular masses β will approximate to $m/2kT$ evaluated for the lighter gas. If β is large, as will happen when both gases are at low temperatures, the expressions (8) and (9) will be required at quite small relative drift velocities. It is noticed that in case 3, $n = 5$, the resistance is independent of the kinetic temperature of the gases.

The more obvious simplifications in one method are those of taking the molecules to be spheres, with a consequently inaccurate cross section, and the assumption that the velocity distribution about the drift velocity of each gas is Maxwellian. The latter assumption is satisfactory if the number of collisions between unlike molecules is much less than that between like molecules. It is hoped that in any case the expressions we have obtained will be useful as a more accurate guide to the resistance in the various applications. In particular, the expressions may be useful in modifying the orders of magnitude in problems of gas motion in astrophysical problems. For example, the possibility of separation of gases of heavy and light elements is of considerable interest, and also the relative motion of gases with widely different temperatures or densities.

Acknowledgments

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